Math 246B Lecture 22 Notes

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1 Picard's Little Theorem and Schottky's Theorem

1.1 Picard's little theorem

Last time, we proved Bloch's theorem:

Theorem 1.1 (A. Bloch)**.** *There exists an absolute constant $\ell > 0$ such that if $f \in \mathrm{Hol}(|z| < 1)$ and $f'(0) = 1$, then the range of $f(D)$ contains an open disc of radius ℓ.*

We can now prove prove Picard's little theorem.[1]

Theorem 1.2 (Picard's little theorem)**.** *Let $f \in \mathrm{Hol}(\mathbb{C})$ be entire and nonconstant. Then the range $f(\mathbb{C})$ omits at most 1 point of $\mathbb{C}$.*

Proof. Let $f \in \mathrm{Hol}(\mathbb{C})$, and assume that f omits 2 distinct values $a, b \in \mathbb{C}$. By composing with an affine transformation, we may assume that $a = 0$, $b = 1$. We will show that f is constant.

We claim that there exists $g \in \mathrm{Hol}(\mathbb{C})$ such that $f(z) = -\exp(i\pi \cosh(2g(z)))$. The function $f \neq 0$ in $\mathbb{C}$, so there exists $F \in \mathrm{Hol}(\mathbb{C})$ such that $e^{2\pi i F} = f$. Moreover, F does not assume integer values, so we can define $\sqrt{F} - \sqrt{F-1} \in \mathrm{Hol}(\mathbb{C})$ which is also nonvanishing. Define g as a holomorphic branch of $\log(\sqrt{F} - \sqrt{F-1})$. Then

$$e^{g} = \sqrt{F} - \sqrt{F-1},$$

$$e^{-g} = \sqrt{F} + \sqrt{F-1}$$

so

$$\cosh(2g) + 1 = 2\cos^2(g) = 2F,$$

which proves the claim.

Let

$$E = \{\pm \underbrace{\log(\sqrt{n} + \sqrt{n-1})}_{=\lambda_n} + im\pi/2 : m \in \mathbb{Z}, n \geq 1\}.$$

[1] This proof is not Picard's original proof. Bloch's theorem came after the original proof.

The points in E form the vertices of a grid of rectangles in $\mathbb{C}$. We claim that $E \cap g(\mathbb{C}) = \varnothing$. If $g(z) = \pm \log(\sqrt{n} + \sqrt{n-1}) + im\pi/2$, then

$$\begin{aligned} 2\cosh(2g(z)) &= e^{im\pi}\left((\sqrt{n}+\sqrt{n-1})^2 + (\sqrt{n}-\sqrt{n-1})^2\right)^2 \\ &= (-1)^m 2(2n-1), \end{aligned}$$

so $f(z) = 1$.

We now claim that g is constant. We have that the height of a rectangle R_n in our grid is $\pi/2$, and the width of R_n is $\lambda_{n+1} - \lambda_n = \log\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}+\sqrt{n-1}}\right) \leq C$ for $n \geq 1$. So there exists some $R_0 > 0$ such that each open disc of radius R_0 meets E. If $g'(a) \neq 0$ for some a, then apply Bloch's theorem to the function $g(a + Rz)/Rg'(a)$ for $|z| < 1$, $R > 0$. The range contains a disc of fixed radius $\ell > 0$ for each $R > 0$, so $g(\mathbb{C})$ contains a disc of radius $R\ell|g'(a)|$. But $g(\mathbb{C}) \cap E = \varnothing$, so $R\ell|g'(a)| \leq R_0$; letting $R \to \infty$, we get a contradiction. □

1.2 Schottky's theorem

Here is a consequence of Bloch's theorem. It will allow us to prove Picard's great theorem.

Theorem 1.3 (Schottky)**.** *For each $0 < \alpha < \infty$ and $0 \leq \beta \leq 1$, there exists a constant $M(\alpha, \beta) > 0$ such that if $f \in \mathrm{Hol}(D)$ omits the values $0, 1$ and $|f(0)| \leq \alpha$, then $|f(z)| \leq M(\alpha, \beta)$ for all $|z| \leq \beta$.*

Proof. We may assume $\alpha \geq 2$. Assume that $1/2 \leq |f(0)| \leq \alpha$. Following the proof of Picard's little theorem, let $F \in \mathrm{Hol}(D)$ be such that $e^{2\pi i F} = f$ in D. Chose the branch of f so that $\mathrm{Re}(F(0)) \in [0, 1]$. Then $e^{-2\pi \,\mathrm{Im}(F(0))} = |f(0)|$, so $|\,\mathrm{Im}(F(0))| \leq (1/2\pi)\log(\alpha)$. We will call $C(\alpha)$ any constant that depends only on α. So $|F(0)| \leq C(\alpha)$. Next, $\sqrt{F} - \sqrt{F-1} \in \mathrm{Hol}(D)$, and $|\sqrt{F(0)} - \sqrt{F(0)-1}| \leq |F(0)|^{1/2} + (|F(0)|+1)^{1/2} \leq C(\alpha)$. Finally, let $g \in \mathrm{Hol}(D)$ be such that $e^g = \sqrt{F} - \sqrt{F-1}$. Choose the branch so that $0 \leq \mathrm{Im}(g(0)) < 2\pi$. We can then control $|\,\mathrm{Re}(g(0))|$. We get a constant $C(\alpha) > 0$ such that if $f(z) = \exp(i\pi\cosh(2g(z)))$, then $|g(0)| \leq C(\alpha)$ if $1/2 \leq |f(0)| \leq \alpha$.

Recall that $g(D) \cap E = \varnothing$, where E is as in the proof of Picard's little theorem. Then there is a number R_0 such that $g(D)$ contains no disc. Let $|z| \leq \beta < 1$, and let

$$\varphi(\zeta) = \frac{g(z + (1-\beta)\zeta)}{(1-\beta)g'(z)}$$

where z is such that $g'(z) \neq 0$. This is holomorphic in $|\zeta| < 1$, and $\varphi'(0) = 1$, so $\varphi(D)$ contains a disc of radius ℓ by Bloch's theorem. So $g(D)$ contains a disc of radius $|ell(1-\beta)|g'(z)|$. So $|g'(z)| \leq R_0/(\ell(1-\beta))$ for $|z| \leq \beta$. By integration, we get uniform control on the function g. □

We will finish the proof next time.